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Abstract

We consider a multicriteria variant of the well known combinatorial minisum facility location problem (median problem) with Pareto and lexicographic optimality principles. Necessary and sufficient conditions of a solution stability in such problems to the initial data perturbations are formulated in terms of binary relations. Numerical examples are given.

Keywords: median problem, facility location problem, multicriteria problem, Pareto optimal solution, lexicographically optimal solution, solution stability, binary relations, initial data perturbations.

1. Introduction

Many problems of design, planning and management are related to the multicriteria nature of the task. In many cases, arising multiobjective models add up to the choice of the "best" (in a certain sense) values of variable parameters from a discrete set of feasible solutions. Therefore recent interest of mathematicians in multicriteria discrete optimization problems keeps very high. It is confirmed by the intensive publishing activity (see, e.g., bibliography [1], which contains 234 references).

While solving practical optimization problems, it is necessary to take into account various kinds of uncertainty due to lack of input data, inadequacy of mathematical models to real processes, rounding off, calculating errors etc. It is known that in many cases initial data as a link between a reality and a model could not be defined explicitly. The initial data are defined with a certain error, generally depend on many parameters and require to be specified during the problem solving process. In practice any problem could not be properly posed and solved without at least implicit use of the results of stability analysis and related issues of parametric analysis. Therefore widespread use of discrete optimization models in the last decades inspired many specialists to investigate various aspects of ill-posed problems theory and, in particular, the stability issues.

Postoptimal and parametric analysis of optimization problems investigate how found solutions behave in response to changes in initial data (problem parameters). In general, the methods to perform both of the analyses are based on analyzing the properties of the point-to-set mapping which specifies the optimality principle of the problem. Such research methods has been studied in great detail and covered in the literature on optimization problems with a continuous set of feasible solutions. Numerous articles are devoted to analysis of conditions when a problem solution possesses a certain property of invariance to the problem parameters perturbations (see, e.g., [2–5]).

The main difficulty while studying stability of discrete optimization problems is discrete models complexity, because even small changes of initial data make a model behave in an unpredictable manner. There are a lot of papers (see, e. g., [6–14]) devoted to analysis of scalar and vector (multicriteria) discrete optimization problems sensitivity to parameters perturbations. The present work continues our investigations of different stability types of discrete optimization problems with various partial criteria and optimality principles (see, e.g., [15–24]). Here the multicriteria variant of the well-known facility location problem (median problem) is considered. Necessary and sufficient conditions of lexicographic and Pareto optima stability under perturbations of initial data are obtained. Numerical examples illustrating obtained results are presented.

2. Basic definitions and notations

Problems of finding the "best" location of equipment and facilities abound in many practical areas. Often such problems are formulated as extreme problems in graphs and networks [25–28]. In the "classical" median problem associated with the location of facilities (medians) it is required to locate facilities so that the sum of all the shortest distances from the nearest facility to the consumers is minimized. Such problems appear often in practice in a variety of forms: locating substations in electric power networks, supply depots in a road distribution network, and many other kinds of facilities. If there are several cost criteria (not only transportation costs) which have to be minimized, then the vector variant of the medians location problem arises. Let us consider this problem in the following formulation.

Let $N_m = \{1, 2, \dots, m\}$ be the set of possible points of facilities (suppliers) location, N_n be consumers (clients) location, $A = (a_{ijk}) \in \mathbb{R}^{m \times n \times s}$ be the matrix of costs a_{ijk} . The cost is connected with client $j \in N_n$ serviced by facility $i \in N_m$ and with criterion $k \in N_s$.

Let on a set T of nonempty subsets of N_m , $T \subset 2^{N_m}$, $|T| \geq 2$, a vector function

$$f(t, A) = (f_1(t, A), f_2(t, A), \dots, f_s(t, A))$$

be defined in the following way:

$$f_k(t, A) = \sum_{j \in N_n} \min_{i \in t} a_{ijk} \rightarrow \min_{t \in T}, \quad k \in N_s.$$

Under the vector (multicriteria) problem of median location we understand both the problem of finding the Pareto set and the problem of finding the lexicographic set. We introduce the sets by using the binary relations of domination $\succ_{P,A}$ and $\succ_{L,A}$ on the set T :

$$t \succ_{P,A} t' \Leftrightarrow f(t, A) \geq f(t', A) \text{ \& } f(t, A) \neq f(t', A),$$

$$t \succ_{L,A} t' \Leftrightarrow \exists r \in N_s (f_r(t, A) > f_r(t', A) \text{ \& } r = \min\{k \in N_s : f_k(t, A) \neq f_k(t', A)\}).$$

Thus, we have the set of Pareto optimal solutions (the Pareto set)

$$P^s(A) = \{t \in T : \forall t' \in T \ (t \not\succ_{P,A} t')\},$$

and the set of lexicographically optimal solutions (the lexicographic set)

$$L^s(A) = \{t \in T : \forall t' \in T \ (t \not\succ_{L,A} t')\}.$$

Here and further the line over a binary relation means a negation of the binary relation

$$t \overline{\succ}_{P,A} t' \Leftrightarrow f(t, A) = f(t', A) \vee \exists r \in N_s (f_r(t, A) < f_r(t', A)),$$

$$t \succ_{L,A} t' \Leftrightarrow (f(t, A) = f(t', A)) \vee$$

$$\vee \exists r \in N_s \ (f_r(t, A) < f_r(t', A) \ \& \ r = \min\{k \in N_s \ (f_k(t, A) \neq f_k(t', A))\}).$$

Thus, the two s -criteria medians location problems are defined: with the Pareto principle of optimality, i.e. the problem $Z_P^s(A)$ of finding the Pareto set $P^s(A)$, and with the lexicographic principle of optimality, i.e. the problem $Z_L^s(A)$ of finding the lexicographic set $L^s(A)$.

Notice that $\emptyset \neq L^s(A) \subseteq P^s(A)$ for any $A \in \mathbf{R}^{m \times n \times s}$ if $1 < |T| < \infty$. Moreover, it is evident that the Pareto set coincides with the lexicographic set in the scalar case ($s = 1$), i.e. $P^1(A) = L^1(A)$ is the set of optimal solutions, $A = (a_{ij}) \in \mathbf{R}^{m \times n}$.

It is known (see, e.g., [29]), that $L^s(A)$ can be defined as the result of solving the sequence of s scalar problems

$$L_k^s(A) = \text{Arg min}\{f_k(t, A) \mid t \in L_{k-1}^s(A)\}, \quad k \in N_s, \quad (1)$$

where $L_0^s(A) = T$, $\text{Arg min}\{\cdot\}$ is the set of all optimal trajectories of the corresponding scalar minimization problem. Hence we have

$$T \supseteq L_1^s(A) \supseteq L_2^s(A) \supseteq \dots \supseteq L_s^s(A) = L^s(A). \quad (2)$$

Particularly in the scalar case ($s = 1$) we get the well-known p -median problem [25–28], i.e. the minisum location problem:

$$\sum_{j \in N_n} \min_{i \in t} a_{ij} \rightarrow \min,$$

$$t \in T, \ |t| = p,$$

where p is an integer number satisfying $1 \leq p \leq m - 1$. Thereby in such a problem it is required to locate p facilities in N_m possible points to minimize the sum of the shortest distances (or transport costs) to any consumer from its nearest facility. Let us mention that in last decades, location models and related optimization methods have become a separate branch of discrete optimization with vast literature, including a number of reviews and monographs. High interest to the mathematical models of the facilities location is connected with wide application, in strategic planning, service network design, choosing an optimal behavior strategy on competitive markets, in standardization and unification of technical facilities and other areas.

We will investigate conditions under which an optimal solution remains optimal under "small" perturbations of vector criterion parameters. Such perturbations are modeled by adding matrix A to matrices of the set

$$\Omega(\varepsilon) = \{A' \in \mathbf{R}^{m \times n \times s} \ : \ \|A'\| < \varepsilon\}.$$

where $\varepsilon > 0$, $\|A'\| = \max\{|a'_{ijk}| : (i, j, k) \in N_m \times N_n \times N_s\}$, $A' = (a'_{ijk})$. The set $\Omega(\varepsilon)$ is called the set of perturbing matrices.

Definition 1. A Pareto optimal solution $t \in P^s(A)$ is called stable if

$$\exists \varepsilon > 0 \quad \forall A' \in \Omega(\varepsilon) \quad (t \in P^s(A + A')).$$

Definition 2. A lexicographically optimal solution $t \in L^s(A)$ is called stable if

$$\exists \varepsilon > 0 \quad \forall A' \in \Omega(\varepsilon) \quad (t \in L^s(A + A')).$$

Thus, the solution is stable if it remains optimal under any "small" independent perturbations of the problem parameters, i.e. the elements of A .

3. Properties

We consider a number of evident properties needed to prove stability criteria for trajectories from $P^s(A)$ and $L^s(A)$.

Directly from the definitions of the binary relations $t \succ_{P,A} t'$ and $t \succ_{L,A} t'$ we have

Property 1. If $t \succ_{P,A} t'$, then $t \succ_{L,A} t'$.

Property 2. If $t \succ_{L,A} t'$, then $t \succ_{P,A} t'$.

Property 3. If $t \succ_{L,A} t'$, then $t \notin L^s(A)$.

Property 4. If $t \succ_{L,A} t'$, then $t' \succ_{L,A} t$.

For any solutions t and t' we define binary relations:

$$\begin{aligned} t \sim_A t' &\Leftrightarrow f(t, A) = f(t', A), \\ t \mid\!-\!_A t' &\Leftrightarrow \forall k \in N_s \quad (t \mid\!-\!_{k,A} t'), \\ t \mid\!-\!_{k,A} t' &\Leftrightarrow \forall j \in N_n \quad (N_{jk}(t, A) \supseteq N_{jk}(t', A)), \end{aligned}$$

where

$$N_{jk}(t, A) = \text{Arg min}\{a_{ijk} : i \in t\},$$

i.e.

$$N_{jk}(t, A) = \{l \in t : \min_{i \in t} a_{ijk} = a_{ljk}\}.$$

Let $t^1, t^2 \in T$. We call t^1 and t^2 equivalent, if $t^1 \sim_A t^2$.

From this notations taking into account the continuity of $f_k(t, A)$ in $\mathbf{R}^{n \times m}$ we obtain

Property 5. For any index $k \in N_s$ the following implication is valid: if $t^0 \mid\!-\!_{k,A} t$, then

$$\exists \varepsilon > 0 \quad \forall A' \in \Omega(\varepsilon) \quad (f_k(t, A + A') \geq f_k(t^0, A + A')).$$

It is easy to see that the next property is valid.

Property 6. If $f(t, A) \geq f(t^0, A)$, then $t^0 \underset{L, A}{\succ} t$.

Combining properties 5 and 6, we obtain the following property.

Property 7. If $t^0 \underset{A}{|}\!-\! t$, then

$$\exists \varepsilon > 0 \quad \forall A' \in \Omega(\varepsilon) \quad (t^0 \underset{L, A+A'}{\succ} t).$$

Property 8. Let $t, t^0 \in T$. If any of the following two conditions holds

$$(i) \quad f_1(t, A) > f_1(t^0, A),$$

$$(ii) \quad \exists r \in N_{s-1} \quad (f_{r+1}(t, A) > f_{r+1}(t^0, A)) \ \& \ \forall k \in N_r \quad (t^0 \underset{k, A}{|}\!-\! t),$$

then we have

$$\exists \varepsilon > 0 \quad \forall A' \in \Omega(\varepsilon) \quad (t \underset{L, A+A'}{\succ} t^0). \quad (3)$$

Proof. If $f_1(t, A) > f_1(t^0, A)$, then by the continuity of $f_k(t, A)$ in $\mathbf{R}^{m \times n}$ we have

$$\exists \varepsilon > 0 \quad \forall A' \in \Omega(\varepsilon) \quad (f_1(t, A + A') > f_1(t^0, A + A')).$$

Hence (3) holds.

Now let condition (ii) hold. Then from $t^0 \underset{k, A}{|}\!-\! t$, $k \in N_r$, and property 5 we get

$$\exists \varepsilon' > 0 \quad \forall A' \in \Omega(\varepsilon') \quad \forall k \in N_r \quad (f_k(t^0, A + A') \leq f_k(t, A + A')). \quad (4)$$

Observe that from the inequality $f_{r+1}(t, A) > f_{r+1}(t^0, A)$ by the continuity of $f_{r+1}(t, A)$ in $\mathbf{R}^{m \times n}$ it follows that

$$\exists \varepsilon'' > 0 \quad \forall A' \in \Omega(\varepsilon'') \quad (f_{r+1}(t, A + A') > f_{r+1}(t^0, A + A')). \quad (5)$$

Assuming $\varepsilon = \min\{\varepsilon', \varepsilon''\}$, we derive (3) from (4) and (5).

4. Necessary and sufficient conditions

For any $t^0 \in T$, denote the set of equivalent solutions

$$Q^s(t^0, A) = \{t \in T : t^0 \underset{A}{\sim} t\}.$$

Theorem 1. A solution $t^0 \in P^s(A)$ is stable if and only if

$$\forall t \in Q^s(t^0, A) \quad (t^0 \underset{A}{|}\!-\! t). \quad (6)$$

It is easy to interpret the theorem for a scalar problem, i.e. for $s = 1$. Let the distance from every client j to every facility i is known. Then the theorem states that for optimal solution t^0 to be stable it is necessary and sufficient that all optimal bindings (in the sense of a proximity by distance) of clients to facilities t^0 remain optimal bindings of all the clients to the facilities of any equivalent trajectory t ($f(t, A) = f(t^0, A)$). It is easy to see from example 7 (see images 1 and 2) given below. Solutions t^1 and t^2 are equivalent and stable in this example.

Proof. Necessity. Let $t^0 \in P^s(A)$ be stable, but (6) do not hold. Then there exists $t \in Q^s(t^0, A)$ such that $t^0 \not\sim_A t$. Therefore there exist $q \in N_n$ and $r \in N_s$ such that $N_{qr}(t^0, A) \not\supseteq N_{qr}(t, A)$. Let $l \in N_{qr}(t, A) \setminus N_{qr}(t^0, A)$. Assume $0 < \alpha < \varepsilon$, and consider the perturbing matrix $A^* = [a_{ijk}^*] \in \Omega(\varepsilon)$ with the elements

$$a_{ijk}^* = \begin{cases} -\alpha, & \text{if } (i, j, k) = (l, q, r), \\ 0 & \text{otherwise.} \end{cases}$$

Taking into account $t^0 \sim_A t$ we obtain

$$\begin{aligned} f_r(t, A + A^*) &= \sum_{j \in N_n} \min_{i \in t} (a_{ijr} + a_{ijr}^*) = a_{lqr} - \alpha + \sum_{j \neq q} \min_{i \in t} a_{ijr} = \\ &= \sum_{j \in N_n} \min_{i \in t} a_{ijr} - \alpha = f_r(t, A) - \alpha < f_r(t, A) = f_r(t^0, A) = f_r(t^0, A + A^*), \\ f_k(t, A + A^*) &= f_k(t, A) = f_k(t^0, A) = f_k(t^0, A + A^*) \text{ for } k \neq r. \end{aligned}$$

From here we derive $t^0 \succ_{P, A+A^*} t$. Thus,

$$\forall \varepsilon > 0 \quad \exists A^* \in \Omega(\varepsilon) \quad (t^0 \notin P^s(A + A^*)).$$

This means that t^0 is not stable. The contradiction.

Sufficiency. Let formula (6) be valid for $t^0 \in P^s(A)$. Let $t \in T$ and $t \neq t^0$. Then the following two cases are possible.

Case 1. $t^0 \sim_A t$. Then (6) implies $t^0 \not\sim_A t$. Using properties 2 and 7 we obtain

$$\exists \varepsilon(t) > 0 \quad \forall A' \in \Omega(\varepsilon(t)) \quad (t^0 \not\supset_{P, A+A'} t). \quad (7)$$

Case 2. $t^0 \not\sim_A t$. Then in view of $t^0 \in P^s(A)$ there exists $r \in N_s$ such that $f_r(t^0, A) < f_r(t, A)$, i.e. $t^0 \not\supset_{P, A} t$. From the continuity of $f_r(t, A)$ in $\mathbf{R}^{m \times n}$ we get (7).

Summarizing the both cases, we conclude that $t^0 \in P^s(A)$ remains efficient in the perturbed problem $Z_P^s(A + A')$ for any $A' \in \Omega(\varepsilon)$, $\varepsilon = \min\{\varepsilon(t) : t \in T\}$. Therefore t^0 is stable.

Theorem 1 is proved.

Theorem 2. A solution $t^0 \in L^s(A)$ is stable if and only if

$$\forall k \in N_s \quad \forall t \in L_k^s(A) \quad (t^0 \mid\!\!\!-\!_{k,A} t). \quad (8)$$

Proof. Necessity. Assume the converse. Let $t^0 \in L^s(A)$ be stable but (8) do not hold. Then

$$\exists r \in N_s \quad \exists t^* \in L_r^s(A) \quad (t^0 \mid\!\!\!-\!_{r,A} t^*).$$

Therefore there exists $q \in N_n$ such that $N_{qr}(t^0, A) \not\supseteq N_{qr}(t^*, A)$. Hence there exists $l \in N_{qr}(t^*, A) \setminus N_{qr}(t^0, A)$. Moreover, since $t^0 \in L^s(A)$, the equality $f_k(t^0, A) = f_k(t^*, A)$ is valid for any $k \in N_r$. Taking into account the said above and setting $0 < \alpha < \varepsilon$ we build the elements of the perturbing matrix $A^0 = (a_{ijk}^0) \in \Omega(\varepsilon)$ by putting

$$a_{ijk}^0 = \begin{cases} -\alpha, & \text{if } (i, j, k) = (l, q, r), \\ 0 & \text{otherwise.} \end{cases}$$

Then we obtain

$$\begin{aligned} f_r(t^*, A + A^0) &= \sum_{j \in N_n} \min_{i \in t^*} (a_{ijr} + a_{ijr}^0) = a_{lqr} - \alpha + \sum_{j \neq q} \min_{i \in t^*} a_{ijr} = \\ &= \sum_{j \in N_n} \min_{i \in t^*} a_{ijr} - \alpha = f_r(t^*, A) - \alpha < f_r(t^*, A) = f_r(t^0, A) = f_r(t^0, A + A^0), \\ f_k(t^*, A + A^0) &= f_k(t^*, A) = f_k(t^0, A) = f_k(t^0, A + A^0), \quad k \in N_{r-1}. \end{aligned}$$

Therefore $t^0 \succ_{L, A+A^0} t^*$. Hence

$$\exists t^* \in T \quad \forall \varepsilon > 0 \quad \exists A^0 \in \Omega(\varepsilon) \quad (t^0 \succ_{L, A+A^0} t^*). \quad (9)$$

Thereby from property 3 it follows that

$$\forall \varepsilon > 0 \quad \exists A^0 \in \Omega(\varepsilon) \quad (t^0 \notin L^s(A + A^0)).$$

This contradicts to the assumption of stability of t^0 .

Sufficiency. Let (8) hold for the lexicographic optimum t^0 of the problem $Z_L^s(A)$. We will show that t^0 is stable. We consider two possible cases. Let $t \in T$.

Case 1. $t \in L_1^s(A)$. At first let $t \in L^s(A)$. Then from (8) and inclusions $L^s(A) \subseteq L_k^s(A)$, $k \in N_s$, we have

$$t^0 \mid\!\!\!-\!_A t.$$

According to property 7 we obtain

$$\exists \varepsilon(t) > 0 \quad \forall A' \in \Omega(\varepsilon(t)) \quad (t^0 \succ_{L, A+A'} t). \quad (10)$$

Now let $t \in L_1^s(A) \setminus L^s(A)$. Then there exists $r = r(t) \in N_s \setminus \{1\}$ such that $t \notin L_r^s(A)$ and $t \in L_k^s(A)$ for $k \in N_{r-1}$. Hence we have

$$\forall k \in N_{r-1} \quad (t^0 \underset{k, A}{\mid} t),$$

$$f_r(t, A) > f_r(t^0, A).$$

Using the said above and property 8(ii), we conclude

$$\exists \varepsilon(t) > 0 \quad \forall A' \in \Omega(\varepsilon(t)) \quad (t \succ_{L, A+A'} t^0).$$

This implies (10) in view of property 4.

Case 2. $t \in T \setminus L_1^s(A)$. Then $f_1(t, A) > f_1(t^0, A)$. Observe that properties 4 and 8(i) imply (10).

Resuming the both cases, we have

$$\forall t \in T \quad \exists \varepsilon(t) > 0 \quad \forall A' \in \Omega(\varepsilon(t)) \quad (t^0 \succ_{L, A+A'} t).$$

Hence $t^0 \in L^s(A)$ is a lexicographic optimum of $Z_L^s(A + A')$ for any $A' \in \Omega(\varepsilon^*)$, $\varepsilon^* = \min\{\varepsilon(t) : t \in T\}$. Consequently, $t^0 \in L^s(A)$ is stable.

Theorem 2 is proved.

5. Corollaries

A number of corollaries follow from theorems 1 and 2.

Corollary 1.

$$Q^s(t^0, A) = \{t^0\}$$

is a sufficient condition for an efficient solution t^0 to be stable.

Corollary 2.

$$L_1^s(A) = \{t^0\}$$

is a sufficient condition for a lexicographic optimum t^0 to be stable.

Further we consider some particular cases.

In the case $p = 1$ ($|t| = 1$ for any $t \in T$) we have the vector 1-median problem. It is easy to see that in this case the sufficient conditions of stability is also necessary. Therefore, the next corollaries follow from theorems 1 and 2.

Corollary 3. If $|t| = 1$ for any $t \in T$, then the equality $Q^s(t^0, A) = \{t^0\}$ is a necessary and sufficient condition of stability of $t^0 \in P^s(A)$.

Corollary 4. If $|t| = 1$ for any $t \in T$, then the equality $L_1^s(A) = \{t^0\}$ is a necessary and sufficient condition of stability of $t^0 \in L^s(A)$.

It is evident that in the case $n = 1$ ($A = (a_{ik}) \in \mathbf{R}^{m \times s}$), the considered problem transforms into the s -criteria combinatorial problem with partial criteria MINMIN:

$$f_k(t, A) = \min_{i \in t} a_{ik} \rightarrow \min_{t \in T}, \quad k \in N_s. \quad (11)$$

For any $k \in N_s$, put

$$N_k(t, A) = \text{Arg min}\{a_{ik} : i \in t\}.$$

We assume $f_k(\emptyset, A_k) = +\infty$.

Next two well-known results follow from theorems 1 and 2.

Corollary 5 [30, 31]. Let $t^0 \in P^s(A)$ be a Pareto optimal solution of the vector problem with partial criteria MINMIN (11). The next statements are equivalent:

- (i) $t^0 \in P^s(A)$ is stable;
- (ii) $\forall k \in N_s \quad \forall t \in Q^s(t^0, A) \quad (N_k(t^0, A) \supseteq N_k(t, A));$
- (iii) $\forall k \in N_s \quad \forall t \in Q^s(t^0, A) \quad (f_k(t \setminus t^0, A) > f_k(t^0, A)).$

Corollary 6 [32]. Let $t^0 \in L^s(A)$ be a lexicographically optimal solution of the vector problem with partial criteria MINMIN (11). The next statements are equivalent:

- (i) $t^0 \in L^s(A)$ is stable;
- (ii) $\forall k \in N_s \quad \forall t \in L_k^s(t^0, A) \quad (N_k(t^0, A) \supseteq N_k(t, A));$
- (iii) $\forall k \in N_s \quad \forall t \in L_k^s(t^0, A) \quad (f_k(t \setminus t^0, A) > f_k(t^0, A)).$

6. Examples

Let us present a number of examples illustrating the results stated above.

Example 1. Let $m = 3$, $n = 3$, $s = 2$, $T = \{t^1, t^2, t^3, t^4\}$, $t^1 = \{1, 2\}$, $t^2 = \{1, 3\}$, $t^3 = \{2, 3\}$, $t^4 = \{3\}$ and

$$A_1 = \begin{pmatrix} 0 & 2 & 1 \\ 1 & 2 & 2 \\ 1 & 0 & 2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}.$$

Then $f(t^1, A) = (3, 2)$, $f(t^2, A) = (2, 3)$, $f(t^3, A) = (3, 2)$, $f(t^4, A) = (3, 3)$.
Therefore $P^2(A) = \{t^1, t^2, t^3\}$, $t^1 \underset{A}{\sim} t^3 \underset{A}{\sim} t^2$. According to corollary 1, t^2 is stable. Observe that

$$N_{11}(t^1, A) = \{1\}, \quad N_{11}(t^3, A) = \{2\}.$$

Hence

$$t^1 \underset{1,A}{\overline{\mid}} t^3, \quad t^3 \underset{1,A}{\overline{\mid}} t^1,$$

i.e.

$$t^1 \underset{A}{\overline{\mid}} t^3 \underset{A}{\overline{\mid}} t^1$$

It follows that both the equivalent efficient solutions t^1 and t^2 are not stable according to theorem 1.

Further we consider stability of equivalent efficient solutions only, because a solution is stable if it is not equivalent to any other solution. At first we give an example of a problem with Pareto optimal solutions $t^1 \underset{A}{\sim} t^2$, one of which is stable and the other is not.

Example 2. Let $m = 2$, $n = 3$, $s = 2$, $T = \{t^1, t^2\}$, $t^1 = \{1, 2\}$, $t^2 = \{1\}$ and

$$A_1 = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \end{pmatrix}.$$

Then $f(t^1, A) = (2, 2)$, $f(t^2, A) = (2, 2)$. Therefore $P^2(A) = \{t^1, t^2\}$ and $t^1 \underset{A}{\sim} t^2$. Further we obtain

$$\begin{aligned} N_{11}(t^1, A) &= N_{11}(t^2, A) = \{1\}, \\ \{1, 2\} &= N_{21}(t^1, A) \supset N_{21}(t^2, A) = \{1\}, \\ \{1, 2\} &= N_{31}(t^1, A) \supset N_{31}(t^2, A) = \{1\}, \\ N_{12}(t^1, A) &= N_{12}(t^2, A) = \{1\}, \\ N_{22}(t^1, A) &= N_{22}(t^2, A) = \{1\}, \\ \{1, 2\} &= N_{32}(t^1, A) \supset N_{32}(t^2, A) = \{1\}. \end{aligned}$$

Hence we conclude

$$\begin{aligned} t^1 \underset{1,A}{\mid} t^2, \quad t^1 \underset{2,A}{\mid} t^2, \\ t^2 \underset{1,A}{\overline{\mid}} t^1, \quad t^2 \underset{2,A}{\mid} t^1. \end{aligned}$$

Therefore

$$t^1 \underset{A}{\mid} t^2 \underset{A}{\overline{\mid}} t^1.$$

According to theorem 1, t^1 is stable and t^2 is not stable.

Now we consider an example where both equivalent Pareto optimal solutions are stable.

Example 3. Let $m = 3$, $n = 3$, $s = 2$, $T = \{t^1, t^2, t^3\}$, $t^1 = \{1, 2\}$, $t^2 = \{1, 3\}$, $t^3 = \{2\}$ and

$$A_1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & 2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 3 & 1 \\ 2 & 3 & 2 \end{pmatrix}.$$

Then $f(t^1, A) = (1, 3)$, $f(t^2, A) = (1, 3)$, $f(t^3, A) = (5, 7)$. Therefore $P^2(A) = \{t^1, t^2\}$, $t^1 \sim_A t^2$. The equalities

$$N_{jk}(t^1, A) = N_{jk}(t^2, A) = \{1\}$$

are valid for any $(j, k) \in N_3 \times N_2$. Therefore

$$t^2 \underset{A}{\vdash} t^1 \underset{A}{\vdash} t^2.$$

Consequently, equivalent efficient solutions t^1 and t^2 are stable by theorem 1.

It should be noticed that example 3 shows that the condition $Q^s(t^0, A) = \{t^0\}$ is not necessary for the efficient solution t^0 to be stable.

Further we consider a number of examples concerned stability of lexicographically optimal solutions.

Example 4. Let $m = 3$, $n = 3$, $s = 2$, $T = \{t^1, t^2, t^3\}$, $t^1 = \{1, 2, 3\}$, $t^2 = \{1, 2\}$, $t^3 = \{1, 3\}$ and

$$A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 1 & 2 \end{pmatrix}.$$

Then $f(t^1, A) = (1, 2)$, $f(t^2, A) = (1, 2)$, $f(t^3, A) = (1, 4)$. Therefore $L_1^2(A) = \{t^1, t^2, t^3\} = T$, $L^2(A) = L_2^2(A) = \{t^1, t^2\}$. Further we obtain

$$N_{11}(t^1, A) = N_{11}(t^2, A) = N_{11}(t^3, A),$$

$$\{1, 2\} = N_{21}(t^1, A) = N_{21}(t^2, A) \supset N_{21}(t^3, A) = \{1\},$$

$$\{1, 2\} = N_{31}(t^1, A) = N_{31}(t^2, A) \supset N_{31}(t^3, A) = \{1\},$$

$$N_{12}(t^1, A) = N_{12}(t^2, A) = \{2\},$$

$$N_{22}(t^1, A) = N_{22}(t^2, A) = \{2\},$$

$$N_{32}(t^1, A) = N_{32}(t^2, A) = \{1, 2\}.$$

Hence the relations

$$\forall k \in N_2 \quad \forall t \in L_k^2(A) \quad (t^1 \mid\!\!\!\overline{}_{k,A} t),$$

$$\forall k \in N_2 \quad \forall t \in L_k^2(A) \quad (t^2 \mid\!\!\!\overline{}_{k,A} t)$$

are valid. Thus, lexicographically optimal solutions t^1 and t^2 are stable by theorem 2.

It should be noticed that example 4 shows that the condition $L_1^s(A) = \{t^0\}$ is not necessary for the lexicographically optimal solution t^0 to be stable.

We consider the problem in which each lexicographically optimal trajectory is not stable.

Example 5. Let $m = 3$, $n = 3$, $s = 2$, $T = \{t^1, t^2, t^3\}$, $t^1 = \{1, 3\}$, $t^2 = \{2, 3\}$, $t^3 = \{1, 2\}$ and

$$A_1 = \begin{pmatrix} 0 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & 2 & 2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 2 & 0 \\ 2 & 0 & 2 \end{pmatrix}.$$

Then $f(t^1, A) = (2, 2)$, $f(t^2, A) = (2, 2)$, $f(t^3, A) = (2, 3)$. Therefore $L_1^2(A) = \{t^1, t^2, t^3\}$, $L^2(A) = L_2^2(A) = \{t^1, t^2\}$. We analyze stability of t^1 and t^2 . Observe that

$$N_{11}(t^1, A) = N_{11}(t^3, A) = \{1\}, \quad N_{11}(t^2, A) = \{3\}.$$

Then we have

$$\exists k = 1 \in N_2 \quad \exists t^2 \in L_1^2(A) \quad (t^1 \mid\!\!\!\overline{}_{1,A} t^2),$$

$$\exists k = 1 \in N_2 \quad \exists t^1 \in L_1^2(A) \quad (t^2 \mid\!\!\!\overline{}_{1,A} t^1).$$

Consequently, both lexicographically optimal solutions t^1 and t^2 are not stable according to theorem 2.

The following example illustrates the situation when stable and not stable trajectories exist in the set $L^s(A)$.

Example 6. Let $m = 2$, $n = 3$, $s = 2$, $T = \{t^1, t^2, t^3\}$, $t^1 = \{1, 2\}$, $t^2 = \{1\}$, $t^3 = \{2\}$ and

$$A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 2 & 1 \end{pmatrix}.$$

Then $f(t^1, A) = (1, 3)$, $f(t^2, A) = (1, 3)$, $f(t^3, A) = (2, 5)$. Therefore $L_1^2(A) = \{t^1, t^2\}$, $L^2(A) = L_2^2(A) = \{t^1, t^2\}$, $t^1 \sim_A t^2$. Further, we obtain

$$N_{11}(t^1, A) = N_{11}(t^2, A) = \{1\},$$

$$\begin{aligned}
\{1, 2\} &= N_{21}(t^1, A) \supset N_{21}(t^2, A) = \{1\}, \\
\{1, 2\} &= N_{31}(t^1, A) \supset N_{31}(t^2, A) = \{1\}, \\
N_{12}(t^1, A) &= N_{12}(t^2, A) = \{1\}, \\
\{1, 2\} &= N_{22}(t^1, A) \supset N_{22}(t^2, A) = \{1\}, \\
N_{32}(t^1, A) &= N_{32}(t^2, A) = \{1\}.
\end{aligned}$$

Hence we conclude

$$\forall k \in N_2 \quad \forall t \in L_k^2(A) \quad (t^1 \mid\!\!\!\! \overline{\mid}_{k,A} t), \quad (12)$$

$$\exists k = 1 \in N_2 \quad \exists t^1 \in L_1^2(A) \quad (t^2 \mid\!\!\!\! \overline{\mid}_{1,A} t^1). \quad (13)$$

Stability of $t^1 \in L^2(A)$ follows from (12). It follows from (13) that $t^2 \in L^2(A)$ is not stable according to theorem 2.

Finally we present an example of the scalar (s=1) 2-median problem.

Example 7. Let $N_3 = \{1, 2, 3\}$ be the possible locations of two facilities, $N_2 = \{1, 2\}$ be the locations of consumers; $T = \{t^1, t^2, t^3\}$, $t^1 = \{1, 2\}$, $t^2 = \{1, 3\}$, $t^3 = \{2, 3\}$ and let the distance (or cost) matrix A is defined as follows

$$A = \begin{pmatrix} 1 & 3 \\ 1 & 4 \\ 2 & 4 \end{pmatrix}.$$

In figures 1 – 3 the locations of facilities are shown by facility numbers in squares and clients numbers in circles. The distances are shown over arcs. According to the problem formulation three ways of facility locations (grey squares) are possible (fig. 1 – 3). Bold arcs are the optimal routes of clients' service.

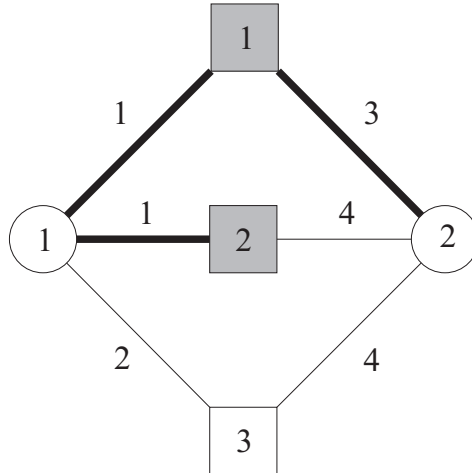


Fig. 1. Solution $t^1 = \{1, 2\}$, $f(t^1, A) = 4$.

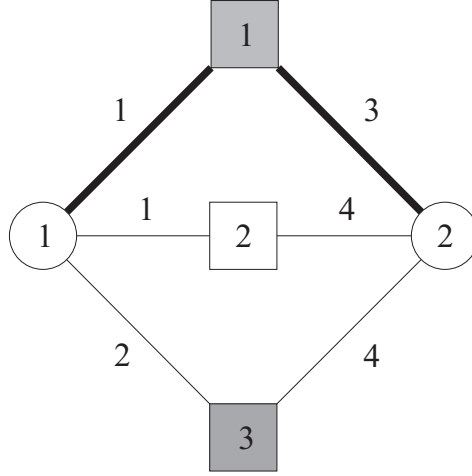


Fig. 2. Solution $t^2 = \{1, 3\}$, $f(t^2, A) = 4$.

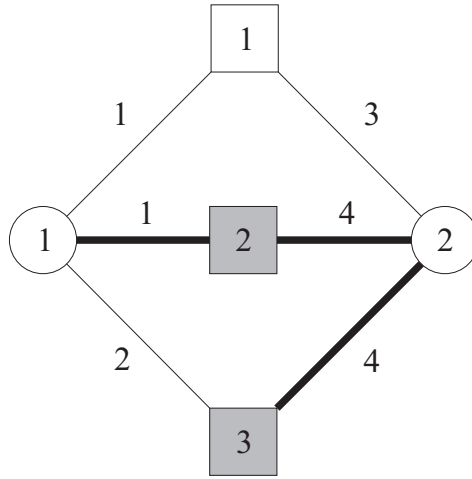


Fig. 3. Solution $t^3 = \{2, 3\}$, $f(t^3, A) = 5$.

Calculating the values $f(t^1, A) = \min\{1, 1\} + \min\{3, 4\} = 1 + 3 = 4$, $f(t^2, A) = \min\{1, 2\} + \min\{3, 4\} = 1 + 3 = 4$, $f(t^3, A) = \min\{1, 2\} + \min\{4, 4\} = 1 + 4 = 5$ we obtain the set of optimal solutions (2-medians)

$$P^1(A) = L^1(A) = \{t^1, t^2\}.$$

The solutions t^1 and t^2 are equivalent.

Further we obtain

$$\{1, 2\} = N_{11}(t^1, A) \supset N_{11}(t^2, A) = \{1\},$$

$$N_{21}(t^1, A) = N_{21}(t^2, A) = \{1\}.$$

Hence we conclude

$$\forall k \in N_1 \quad \forall t \in L_k^1(A) \quad (t^1 \mid\!\!\! \text{---} t)_{k,A},$$

$$\exists k = 1 \in N_1 \quad \exists t^1 \in L_1^1(A) \quad (t^2 \overline{\underset{1,A}{\mid}} t^1).$$

Therefore by virtue of theorem 1 t^1 is stable and t^2 is not stable.

It is easy to see without applying theorem 1, that t^2 is not stable. Let us consider the perturbing matrix for $\varepsilon > 0$

$$A^* = \begin{pmatrix} 0 & 0 \\ -\alpha & 0 \\ 0 & 0 \end{pmatrix},$$

where $0 < \alpha < \varepsilon$. Then we obtain

$$f(t^1, A + A^*) = \min\{1, 1 - \alpha\} + \min\{3, 4\} = 4 - \alpha,$$

$$f(t^2, A + A^*) = \min\{1, 2\} + \min\{3, 4\} = 4.$$

Therefore the relation

$$t^2 \underset{A+A^*}{\succ} t^1$$

is true. Consequently,

$$\forall \varepsilon > 0 \quad \exists A^* \in \Omega(\varepsilon) \quad (t^2 \notin P^1(A + A^*)),$$

i.e. t^2 is not stable.

7. Conclusions

In the present article we investigate stability of solutions of the vector variant of the extreme combinatorial median location problem. Medians can be understood as facility sites which serve customers demands with minimized total delivery costs. Under stability we understand existence of a neighborhood in space of initial data such that an optimal solution preserves optimality in any perturbed problem with initial data from this neighborhood. Necessary and sufficient conditions of Pareto optimal and lexicographically optimal solutions stability are formulated in the terms of binary relations. Similar results were obtained previously in papers [33, 34], where necessary and sufficient conditions on one type of stability of the Pareto set and the lexicographic set (not a single solution) were investigated for the vector combinatorial center and median location problems. Theorems which are similar to theorems 1 and 2 were proved for the multicriteria combinatorial center location problem in [35].

The results and apparatus presented in the current article can be used to create and analyze algorithms for a vector location problems solving under different types of uncertainties.

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